Lecture #16.

Last time:

Prop: Let $X: S \to \mathbb{R}$ be a DRV. Then
$$E[X] = \sum_{s \in S} X(s)p(s).$$

We saw a proof of this fact at the end of the last lecture. Using this, we have the following nice corollary:

Cor: Let $X_1, \dots, X_n$ be DRV's, $X_i : S \to \mathbb{R}$. Then for constants $c_0, c_1, \dots, c_n \in \mathbb{R}$,
$$E[c_0 + c_1 X_1 + \dots + c_n X_n] = c_0 + c_1 E[X_1] + \dots + c_n E[X_n].$$

Proof: Let $Z = c_0 + c_1 X_1 + \dots + c_n X_n$. Then
$$E[Z] = \sum_{s \in S} Z(s)\, p(s)$$
$$= \sum_{s \in S} (c_0 + c_1 X_1(s) + \dots + c_n X_n(s))\, p(s)$$

$$= \sum_{s \in S} c_0 p(s) + c_1 X_1(s) p(s) + \dots + c_n X_n(s) p(s)$$

$$= c_0 \underbrace{\sum_{s \in S} p(s)}_{=1} + c_1 \underbrace{\sum_{s \in S} X_1(s) p(s)}_{E[X_1]} + \dots + c_n \underbrace{\sum_{s \in S} X_n(s) p(s)}_{E[X_n]}.$$

$$= c_0 + c_1 E[X_1] + \dots + c_n E[X_n].$$

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(Note: This is a little more general than Cor 9.2 in the textbook).

<u>Ex:</u> n fair dice are rolled. What is the expected sum? Let X_i = the value of the ith die.

Then $X = X_1 + \dots + X_n$.

$$E[X] = E[X_1] + \dots + E[X_n]$$

$$= \underbrace{3.5 + \dots + 3.5}_{n \text{ times}} = n \cdot (3.5)$$

<u>Ex:</u> Find the total # of successes that result from n Bernoulli trials if the prob. of success for the ith trial is p_i.

Let $X_i = \begin{cases} 1 & \text{if ith trial is a success} \\ 0 & \text{o/w} \end{cases}$

Then $X = X_1 + X_2 + \dots + X_n$.

and $E[X] = \sum_{i=1}^{n} p_i$

As a particular example, if the trials are independent each with prob. p of success, then $X = Bin(n,p)$ and

$$E[X] = \sum_{i=1}^{n} p = np = E[Bin(n,p)]$$

This also gives the expected value of the hypergeometric random var. If we choose a sample of size n from a pool of N objects, of which m are considered successes, then for any particular trial, the probability of success is $\frac{m}{N}$, giving

$$E[X] = n\,E[X_1] = n\frac{m}{N}.$$

Properties of Cumulative Dist. Functions (4.10).

Let X be a RV. Recall that the cumulative distribution F(x) of X is the function defined by

$$F(x) = P(X \leq x).$$

A c.d.f. has the following properties:

1) $\forall a,b,\ a<b \rightarrow F(a) \leq F(b)$ (non decreasing)

2) $\lim_{b \to \infty} F(b) = 1$.

3) $\lim_{b \to -\infty} F(b) = 0$

4) Right Continuity: for any b and any <u>decreasing</u> sequence $\{b_i\}$ (so $i < j \Rightarrow b_j < b_i$) such that $b_i \to b$, we have $\lim_{i \to \infty} F(b_i) = F(b)$.

We leave these properties as an exercise/see the textbook.

- Knowing $F(x)$ for a random variable X, essentially tells you everything about X.

- For any $a \leq b$, $P(a < X \leq b) = F(b) - F(a)$.
- This is because we can always write

$$\{X \leq b\} = \underbrace{\{X \leq a\} \cup \{a < X \leq b\}}_{\text{mutually disjoint sets, so}}.$$

$$\underbrace{P(X \leq b)}_{F(b)} = \underbrace{P(X \leq a)}_{F(a)} + P(a < X \leq b)$$

$$\Rightarrow F(b) - F(a) = P(a < X \leq b).$$

To compute $P(X < b)$ we may use continuity (not the right continuity of $F(x)$, but continuity of prob. from chapter 2).

$$P(X < b) = P\left(\lim_{n\to\infty} \{X \le b - 1/n\}\right)$$
$$= \lim_{n\to\infty} P(X \le b - 1/n)$$
$$= \lim_{n\to\infty} F(b - 1/n).$$

Note that $P(X<b)$ is not necessarily $F(b)$.

<u>Example:</u> Suppose that X is an RV with cdf given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x < 1 \\ \frac{2}{3} & 1 \le x < 2. \\ \frac{11}{12} & 2 \le x < 3. \\ 1 & 3 \le x. \end{cases}$$

What is $p\{X<3\}$?

$$P(X < 3) = \lim_{n\to\infty} \{X \le 3 - 1/n\} = \lim_{n\to\infty} \underbrace{F(3-1/n)}_{= \frac{11}{12}.}$$
$$= \lim_{n\to\infty} \frac{11}{12} = \frac{11}{12}.$$

What is $P(X=1)$?

$$P(X=1) = P(X \le 1) - P(X < 1)$$
$$= F(1) - \lim_{n\to\infty} P(X \le 1 - 1/n)$$

$$= F(1) - \lim_{n \to \infty} F(1 - 1/n)$$

$$= \frac{2}{3} - \lim_{n \to \infty} \frac{1 - 1/n}{2} = \frac{2}{3} - \frac{1}{2} = 1/6.$$

What is $P(X > 1/2) = ?$

$\hookrightarrow P(X > 1/2) = 1 - P(X \leq 1/2)$

$= 1 - F(1/2) = \frac{3}{4}.$

what is $P(2 < X \leq 4) = F(4) - F(2).$

$= 1/$